Applied Mathematics for Engineers **Final Exam**		
Code : *MAT 210* Acad.Year : *2018-19* Semester : *Fall* Date : *08.01.2019* Time : *9:00* Duration : *110 min*	Last Name : Name : Solutions Student ID : Signature : 6 QUESTIONS ON 5 PAGES TOTAL 100 POINTS	# :

P1. (22)	P2. (24)	P3. (24)	P4. (20)	P5. (10)		Total. (100)

1. *(11×2=22pts)* Indicate whether a given statement is **TRUE** or **FALSE** by circling your answer. No explanations are required.

Point values are: Incorrect=0pts, Blank=1pt, Correct=2pts.

TRUE / $\boxed{\textbf{FALSE}}$ The eigenvalues of $M^{-1}K$ are the angular frequencies for the fundamental oscillations of spring systems.

Angular frequency $\omega = \sqrt{\lambda}$

TRUE / $\boxed{\textbf{FALSE}}$ The complex Fourier coefficients of $f(t)$ are given by

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{kit}\, dt$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ki\cdot t}$$

TRUE / $\boxed{\textbf{FALSE}}$ The complex Fourier coefficients c_k determine the function $f(t)$ by

$$f(t) = \sum_{k=0}^{\infty} c_k\, e^{kit}$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{ki\cdot t}$$

TRUE / $\boxed{\textbf{FALSE}}$ If the **odd** Fourier coefficients of $f(t)$ are 0 then $f(t)$ is an **even** function.

Odd coefficients $= 0 \Rightarrow f(t)$ is 2-periodic $\quad f(t) = f(t+\pi)$

TRUE / $\boxed{\textbf{FALSE}}$ If the complex Fourier coefficients of $f(t)$ are **real** then $f(t)$ is **odd**.

$c_k = \frac{a_k - b_k i}{2}$ Real $\Rightarrow b_k = 0 \Rightarrow f(t)$ is even

$\boxed{\textbf{TRUE}}$ / **FALSE** For roots of unity, $\omega_{12}^3 = -\overline{\omega_4}$.

$\omega_{12}^3 = \omega_4 = i$ & $\overline{\omega_4} = -i$

TRUE / $\boxed{\textbf{FALSE}}$ The complex Fourier coefficients c_k determine the vector $\mathbf{f}$ by

$$f_n = \sum_{k=0}^{N} c_k\, (\omega_N^n)^k$$

$$f_n = \sum_{k=0}^{N-1} c_k\, (\omega_N^n)^k$$

TRUE / $\boxed{\textbf{FALSE}}$ The convolution $(\mathbf{f} \circledast \mathbf{g})$ is defined even when $\mathbf{f}$ and $\mathbf{g}$ have different lengths.

cyclic convolution requires same length

$\boxed{\textbf{TRUE}}$ / **FALSE** The convolution $(\mathbf{f} * \mathbf{g})$ can have a different length than $\mathbf{f}$.

infinite convolution implicitly pads vectors with 0

TRUE / $\boxed{\textbf{FALSE}}$ Convolutions satisfy the formula $\mathcal{F}_k\{\mathbf{f} * \mathbf{g}\} = N\,\mathcal{F}_k\{\mathbf{f}\}\,\mathcal{F}_k\{\mathbf{g}\}$.

only for cyclic $\mathcal{F}_k\{f \circledast g\} = N\,\mathcal{F}_k\{f\}\,\mathcal{F}_k\{g\}$

TRUE / $\boxed{\textbf{FALSE}}$ Convolutions satisfy the formula $\mathcal{F}\{\mathbf{f} \cdot \mathbf{g}\} = \mathcal{F}\{\mathbf{f}\} \circledast \mathcal{F}\{\mathbf{g}\}$.

$\mathcal{F}\{f\cdot g\} = N\,\mathcal{F}\{f\} \circledast \mathcal{F}\{g\}$

2. *(4×3=12pts)* Suppose $f(t) = 2\cos(3t) - \sin(3t) - 3\sin(6t)$

(A) Write all non-zero **real** Fourier coefficients of $f(t)$.

$$\boxed{a_3 = 2 \quad b_6 = -3 \quad b_3 = -1 \quad (\text{all others} = 0)}$$

(B) Write all non-zero **real** Fourier coefficients of $f(2t)$.

$f(2t) = 2\cos(6t) - \sin(6t) - 3\sin(12t)$

$$\boxed{a_6 = 2 \quad b_{12} = -3 \quad b_6 = -1 \quad (\text{all others} = 0)}$$

(C) Write all non-zero **real** Fourier coefficients of $f(t + \pi/3)$.

$f(t+\pi/3) = 2\cos(3t+\pi) - \sin(3t+\pi) - 3\sin(6t+2\pi)$

$= -2\cos(3t) + \sin(3t) - 3\sin(6t)$

$$\boxed{a_3 = -2 \quad b_6 = -3 \quad b_3 = 1 \quad (\text{all others} = 0)}$$

(D) Write all non-zero **real** Fourier coefficients of $\frac{d}{dt}f(t)$.

$\frac{d}{dt} f(t) = -6\sin(3t) - 3\cos(3t) - 18\cos(6t)$

$$\boxed{b_3 = -6 \quad a_6 = -18 \quad a_3 = -3 \quad (\text{all others} = 0)}$$

3. *(4×3=12pts)* Suppose $g(t) = (1+i)e^{-3it} - ie^{-it} + ie^{it} + (1-i)e^{3it}$

(A) Write all non-zero **complex** Fourier coefficients of $g(t)$.

$$\boxed{c_{-3} = (1+i) \quad c_3 = (1-i) \quad c_{-1} = -i \quad c_1 = i \quad (\text{all others} = 0)}$$

(B) Write all non-zero **real** Fourier coefficients of $g(t)$

$a_1 = 2\,\text{Real}(c_1) = 0$

$b_1 = -2\,\text{Im}(c_1) = -2$

$a_3 = 2\,\text{Real}(c_3) = 2$

$b_3 = -2\,\text{Im}(c_3) = 2$

$$\boxed{b_1 = -2 \quad a_3 = 2 \quad b_3 = 2}$$

(C) Write all non-zero **complex** Fourier coefficients of $g(2t)$.

multiply index by 2

$$\boxed{c_{-6} = (1+i) \quad c_6 = (1-i) \quad c_{-2} = -i \quad c_2 = i}$$

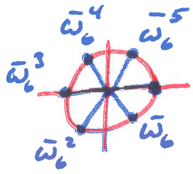
(D) Write all non-zero **complex** Fourier coefficients of $g(t + \pi/2)$.

Recall: $e^{ki(t+\pi/2)} = e^{kit} \cdot e^{ki\cdot\pi/2}$

$e^{k\pi/2\, i} = \begin{cases} \pm 1 \\ \pm i \end{cases}$ depending on k

$$\boxed{c_{-3} = i(1+i) = -1+i \quad c_3 = -1-i \quad c_{-1} = (-i)(-i) = -1 \quad c_1 = -1}$$

4. *(24pts)* Compute the discrete Fourier transform of $\mathbf{f} = \begin{array}{c|l} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} & \begin{matrix} = f_0 \\ = f_1 \\ = f_2 \\ = f_3 \\ = f_4 \\ = f_5 \end{matrix} \end{array}$



length = 6

$\bar{\omega}_6 = \frac{1}{2}(1-\sqrt{3}i)$

$\bar{\omega}_6^2 = \frac{1}{2}(-1-\sqrt{3}i)$

$\bar{\omega}_6^3 = -1$

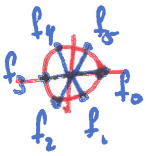
$\bar{\omega}_6^4 = -\bar{\omega}_6$

$\bar{\omega}_6^5 = -\bar{\omega}_6^2$

$\boxed{c_0}$

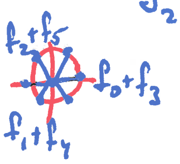
$$\mathcal{F}_0\{\underline{f}\} = \frac{1}{6}\left[0+2+0+0+2+0\right] = \boxed{4/6}$$

$\boxed{c_1}$



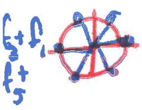
$$\mathcal{F}_1\{\underline{f}\} = \frac{1}{6}\left[(f_0-f_3) + (f_1-f_4)\bar{\omega}_6 + (f_2-f_5)\bar{\omega}_6^2\right]$$
$$= \frac{1}{6}\left[0 + 0 + 0\right] = \boxed{0}$$

$\boxed{c_2}$



$$\mathcal{F}_2\{\underline{f}\} = \frac{1}{6}\left[(f_0+f_3) + (f_1+f_4)\bar{\omega}_6^2 + (f_2+f_5)\bar{\omega}_6^4\right]$$
$$= \frac{1}{6}\left[0 + 4\cdot\frac{1}{2}(-1-\sqrt{3}i) + 0\right] = \boxed{\frac{1}{6}(-2-2\sqrt{3}i)}$$

$\boxed{c_3}$



$$\mathcal{F}_3\{\underline{f}\} = \frac{1}{6}\left[(f_0+f_2+f_4) - (f_1+f_3+f_5)\right]$$
$$= \frac{1}{6}\left[2 - 2\right] = \boxed{0}$$

$\boxed{c_4}$

$$\mathcal{F}_4\{\underline{f}\} = \overline{\mathcal{F}_2\{\underline{f}\}} = \boxed{\frac{1}{6}(-2+2\sqrt{3}i)}$$

$\boxed{c_5}$

$$\mathcal{F}_5\{\underline{f}\} = \overline{\mathcal{F}_1\{\underline{f}\}} = \boxed{0}$$

$$\boxed{\mathcal{F}\{\underline{f}\} = \frac{1}{6}\begin{bmatrix} 4 \\ 0 \\ -2-2\sqrt{3}i \\ 0 \\ -2+2\sqrt{3}i \\ 0 \end{bmatrix}}$$

(Note: There are multiple correct ways to do this computation...)

5. *(4+16=20pts)* The following parts are about different stages of the fast Fourier transform.

(A) If $\mathbf{f} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix}$, write the vectors $\mathbf{f_e}$ and $\mathbf{f_o}$ used for the fast Fourier transform algorithm.

$$f = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix} \qquad f_{even} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad f_{odd} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(B) Suppose that a different vector $\mathbf{g}$ with length 12 has coefficients

$\mathcal{F}_2(\mathbf{g_e}) = 2 + 2i$ and

$\mathcal{F}_2(\mathbf{g_o}) = 4i.$

Compute the following coefficients of $\mathbf{g}$. (Simplify your answers to the form $a + bi$.)

$$\bar{\omega}_{12}^{2} = \bar{\omega}_6 = \tfrac{1}{2}(1-\sqrt{3}i) \qquad \bar{\omega}_{12}^{2}\,\tilde{\mathcal{F}}_2\{g_o\} = \tfrac{1}{2}(1-\sqrt{3}i)\cdot[4i] = 2(\sqrt{3}+i)$$

$$\mathcal{F}_2(\mathbf{g}) = \frac{1}{2}\left[(2+2i) + 2(\sqrt{3}+i)\right] = \boxed{(1+\sqrt{3}) + 2i}$$

$$\mathcal{F}_4(\mathbf{g}) = \boxed{(1-\sqrt{3})}$$

conjugate

change sign

conjugate

$$\mathcal{F}_8(\mathbf{g}) = \frac{1}{2}\left[(2+2i) - 2(\sqrt{3}+i)\right] = \boxed{(1-\sqrt{3})}$$

$$\mathcal{F}_{10}(\mathbf{g}) = \boxed{(1+\sqrt{3}) - 2i}$$

6. *(2×5=10pts)* Compute the following.

(A) The infinite convolution $\begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix} * \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$

		4	-3	5
		1	3	-2
		-8	6	-10
	12	-9	15	
4	-3	5		
4	9	-12	21	-10

$\to \boxed{\begin{bmatrix} -10 \\ 21 \\ -12 \\ 9 \\ 4 \end{bmatrix}}$

(B) The cyclic convolution $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \circledast \begin{bmatrix} -4 \\ -1 \\ 3 \end{bmatrix}$

		2	-3	1
		3	-1	-4
		-8	12	-4
	-2	3	-1	-2
6	-9	3	6	-9
		-2	17	-15

$\to \boxed{\begin{bmatrix} -15 \\ 17 \\ -2 \end{bmatrix}}$